MATH 3235 Probability Theory

11/17/22

$X_i \quad i = 1 \ldots N \quad$ i. i. d. r.v.

$$\mathbb{E}(X_i) = 0 \qquad var(X_i) = 1$$

$$\sum_{i=1}^{N} X_i = S_N$$

S_N is typically of the order of $\sqrt{N}$

$$\mathbb{P}(|S_N| \geq a N^{\alpha}) \xrightarrow[N\to\infty]{} 0 \qquad \alpha > \frac{1}{2}$$

$$\mathbb{P}(|S_N| \leq a N^{1/2}) \xrightarrow[N\to\infty]{} C$$

$$\mathbb{P}(|S_N| \leq a N^{\alpha}) \to 1 \qquad \alpha < \frac{1}{2}$$

What is the probability that $S_N \geq aN$

$$\mathbb{P}(S_N \geq aN) = \mathbb{P}\left(e^{\tau S_N} \geq e^{\tau a N}\right) \leq$$

$$\leq \frac{\mathbb{E}(e^{\tau S_N})}{e^{\tau a N}} =$$

$$= \frac{\mathbb{E}(e^{t X_1})^N}{(e^{at})^N} = \left(\frac{M_X(t)}{e^{at}}\right)^N$$

$$\mathbb{P}(S_N \geq aN) \leq \inf_t \left(M_X(t)\, e^{-at}\right)^N$$

$M_X(t)$ exists in a neighboor of 0, $(-\delta, \delta)$.

Chernoff Inequality.

$$\frac{1}{N} \log \mathbb{P}(S_N > aN) \leq - \sup_t [at - \Lambda(t)]$$

$$\Lambda(t) = \log M_X(t)$$

$$\Lambda^*(a) = - \sup [at - \Lambda(t)]$$

Fenchel-Legendre Transform.

$$\frac{1}{N} \log \mathbb{P}(S_N > aN) \leq - \Lambda^*(a)$$

Kramer Theorem

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}(S_N > aN) = - \Lambda^*(a)$$

$$\Lambda^*(a) > 0$$

$$\Lambda(0) = \log M(0) = 0$$

$$\Lambda'(0) = \frac{M'(0)}{M(0)} = \mathbb{E}(X) = 0$$

$$\Lambda''(t) = \frac{M(t)\,M''(t) - M'(t)^2}{M(t)^2} =$$

$$= \left(\mathbb{E}(e^{tX})\,\mathbb{E}(X^2 e^{tX}) - \mathbb{E}(Xe^{tX})^2 \right) / M(t)^2$$

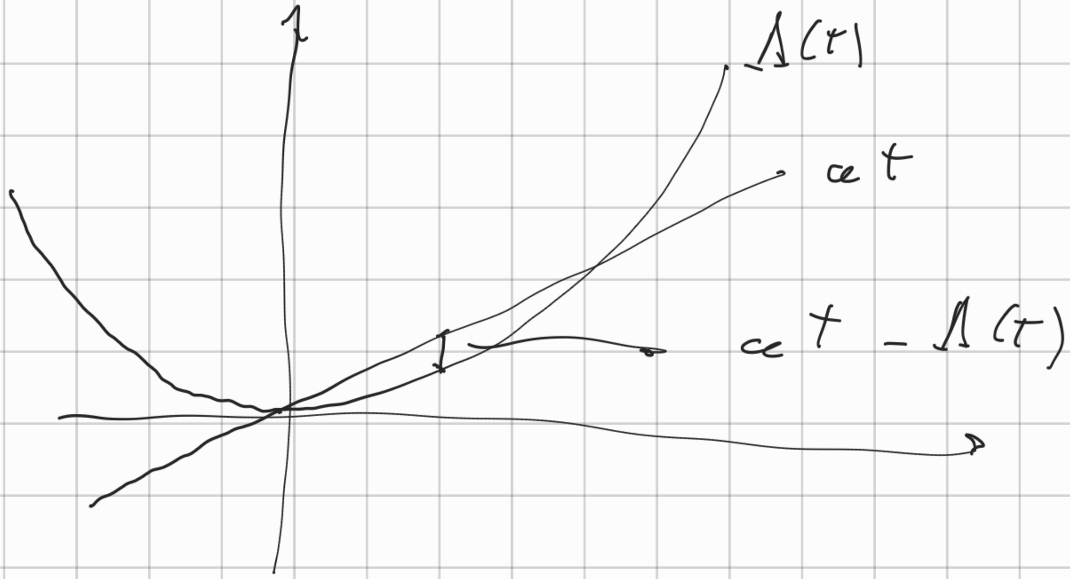
$$Y = X e^{\frac{1}{2}tX} \qquad Z = e^{\frac{1}{2}tX}$$

$$\mathbb{E}(YZ) = \mathbb{E}(Xe^{tX})$$

$$\mathbb{E}(YZ)^2 \leq \mathbb{E}(Y^2)\,\mathbb{E}(Z^2) = \mathbb{E}(X^2 e^{tX})\,\mathbb{E}(e^{tX})$$

$$\Lambda''(t) > 0$$

$\Lambda(t)$ is a convex function with a minimum in 0.



for t small enoog

$$at - \Lambda(t) > 0$$

$$\Downarrow$$

$$\sup_t \left(at - \Lambda(t)\right) > 0$$

Convergence in Probability

Convergence in Distribution.

C. in Prob $\Rightarrow$ C. in Dist.

Theorem 1.

if $Z_n \to Z$ in prob.

$$\Downarrow$$

$$Z_n \Rightarrow Z$$

Theorem 2:

if $Z_n \Rightarrow c$

$$\Downarrow$$

$Z_n \to c$ in prob.

$F_n(z) = \mathbb{P}(Z_n \leq z)$

$F(z) = \mathbb{P}(Z \leq z)$

Goal

$F(z-\varepsilon) + \text{corr} \leq F_n(z) \leq F(z+\varepsilon) + \text{corr}.$

where corr $\to 0$ as $n \to \infty$.

$$F_n(z) = \mathbb{P}(Z_n \leq z) =$$

$$\mathbb{P}(Z_n \leq z \;\&\; Z \leq z+\varepsilon) +$$

$$\mathbb{P}(Z_n \leq z \;\&\; Z > z+\varepsilon) \leq$$

$$\leq \mathbb{P}(Z \leq z+\varepsilon) +$$

$$\mathbb{P}(Z - Z_n > \varepsilon)$$

$$\leq F(z+\varepsilon) + \mathbb{P}(|Z - Z_n| > \varepsilon)$$

$$F_n(z) \leq F(z+\varepsilon) + \underbrace{\mathbb{P}(|Z - Z_n| > \varepsilon)}_{\to 0 \quad n \to \infty}$$

$$F(z-\varepsilon) = \mathbb{P}(Z \leq z-\varepsilon) =$$

$$\mathbb{P}(Z \leq z-\varepsilon \ \& \ Z_n \leq z) +$$

$$\mathbb{P}(Z \leq z-\varepsilon \ \& \ Z_n > z)$$

$$\leq \mathbb{P}(Z_n \leq z) + \mathbb{P}(Z_n - Z > \varepsilon)$$

$$F(z-\varepsilon) \leq F_n(z) + \mathbb{P}(|Z_n - Z| > \varepsilon)$$

$$F(z-\varepsilon) - \mathbb{P}(|Z_n - Z| > \varepsilon) \leq F_n(z) \leq F(z+\varepsilon) + \mathbb{P}(|Z_n - Z| > \varepsilon)$$

$$F(z+\varepsilon) + \mathbb{P}(|Z_n - Z| > \varepsilon) \xrightarrow[n \to \infty]{}$$

$$F(z+\varepsilon) \xrightarrow[\varepsilon \to 0]{} F(z)$$

$$F(z-\varepsilon) - \mathbb{P}(|Z_n - Z| > \varepsilon) \xrightarrow[n \to \infty]{} F(z-\varepsilon)$$

$$\xrightarrow[\varepsilon \to 0]{} F(z)$$

if z is such that $F(z)$ is

continuoo the

$$F_n(z) \underset{n \to \infty}{\to} F(z)$$

$$z_n \Rightarrow z.$$

$$F_n(z) \Rightarrow C \quad \text{as} \quad n \to \infty$$

$$F_n(z) \to \begin{cases} 0 & z < C \\ 1 & z > C \end{cases}$$

$$\mathbb{P}(|z_n - C| > \varepsilon) =$$

$$\mathbb{P}(z_n < C - \varepsilon) + \mathbb{P}(z_n > C + \varepsilon)$$

$$= F_n(C - \varepsilon) + 1 - F_n(C + \varepsilon) \to$$

$$0 + 1 - 1 = 0$$

$X_i \qquad \mathbb{E}(X) = \mu$

$$T_N = \frac{1}{N}\sum_{i=1}^{N} X_i \qquad \phi_X(t)$$

$$\phi_X(t) = \left[1 + \frac{it\mu}{N} + O\left(\frac{t^2}{N^2}\right)\right]$$

$$\phi_{T_N} = \left(1 + \frac{iT\mu}{N} + O\left(\frac{t^2}{N^2}\right)\right)^N \xrightarrow[N\to\infty]{} e^{iT\mu}$$

char funct. of the constant μ.

$$T_N \Rightarrow \mu \quad \Longrightarrow \quad T_N \xrightarrow{P} \mu.$$

if The var is not finite

$$\phi_{T_N}(t) = \left(1 + \frac{i\mu t}{N} + o\left(\frac{T}{N}\right)\right)^N$$